On spin chains, charges and anomalies

1994 J. Phys. A: Math. Gen. 27 3979

(http://iopscience.iop.org/0305-4470/27/11/043)

J. Phys. A: Math. Gen. 27 (1994) 3979–3987. Printed in the UK

On spin chains, charges and anomalies

Harald Grosse, Walter Maderner† and Christian Reitberger†
Institut für Theoretische Physik, Universität Wien, Boltzmanng.5, 1090 Wien, Austria

Received 7 June 1993

Abstract. We discuss temperature effects on anomalies. We deal with an anomaly in a spin-chain model presented in [1], where, in [2], anomaly meltdown was found. We present a slightly different treatment, that to us seems more plausible from a physical point of view, which shows anomaly persistence. We also discuss a similar problem for $U(1)$ Schwinger terms in local current algebras.

1. Introduction

Dealing with finite-temperature models, one usually has many more possibilities to implement symmetries than in the zero-temperature case, which can lead to ambiguities concerning observable quantities. This problem should be resolvable on physical grounds. Recently there has been some interest on temperature effects on anomalies. In this paper we would like to emphasize some points which seem important to us, but might be well known to specialists.

We revisit the anomaly between translations and time evolution in the infinite spin-$\frac{1}{2}$ chain discussed in [1]. We present a different implementation of translations at finite temperature and argue that our implementation is more favourable on physical grounds than that presented in [2]. But then the anomaly does not vanish. Instead it takes values in the commutant of the observable algebra and is thus an example of a non-commutative cocycle [1]. As the temperature approaches zero we recover the usual phase factor.

We also discuss local $U(1)$-current algebras, where a similar problem concerning the (non-)vanishing of Schwinger terms occurs.

2. The spin chain

To fix the notation we review first the anomaly at zero temperature. Let $\mathcal{A}$ be the inductive limit C^* algebra of the local net $\{\mathcal{A}(\Lambda) : \Lambda \subseteq \mathbb{Z}; |\Lambda| < \infty\}$. Where $\mathcal{A}(\Lambda) := \bigotimes_{k\in\Lambda} M_2(\mathbb{C})_k$ and $M_2(\mathbb{C})$ denotes the algebra of 2×2 matrices. Let $A_k := \ldots 1 \otimes A \otimes 1 \ldots$($k$th place), and let σ^l be the lth Pauli matrix. Then the group $\mathbb{Z}$ of translations is represented on $\mathcal{A}$ by *-automorphisms τ_n which are uniquely determined by the assignment

$$A_k \mapsto \tau_n(A_k) := A_{k+n}. \tag{1}$$

† Supported in part by the 'Fonds zur Förderung der wissenschaftlichen Forschung in Österreich' under contract no P8916-PHY.

The time evolution of non-interacting spins in a constant magnetic field $B(n) = B$ is given by the one-parameter group of *-automorphisms

$$A_k \mapsto \alpha_t(A_k) := \exp\left(\mathrm{i}t\frac{M}{2}\sigma_k^3\right) A_k \exp\left(-\mathrm{i}t\frac{M}{2}\sigma_k^3\right) \tag{2}$$

where $M = \mu B$, μ =magnetic moment. Since the external field is translation invariant, we observe $\tau_n \circ \alpha_t = \alpha_t \circ \tau_n$. The spin-flip automorphism γ on the negative half-line defined by

$$A_k \mapsto \gamma(A_k) := \Theta(k)A_k + (1 - \Theta(k))\sigma_k^1 A_k \sigma_k^1 \tag{3}$$

reverses the direction of the external field for all spins on the negative axis. Here we have put $\Theta(k) = 1$ if $k \geqslant 0$ and 0 otherwise. As a consequence, the time evolution α^s of non-interacting spins in the soliton external field $B(k) = \mathrm{sgn}(k)B$; $\mathrm{sgn}(k) := 2\Theta(k) - 1$, is obtained by conjugation of α with γ; that is, $\alpha_t^s(A_k) := \gamma \circ \alpha_t \circ \gamma(A_k)$. Since the external field is not constant, α_t^s commutes with τ_n only up to an inner automorphism. The positive linear form

$$\prod_{l=1}^{n} A_{k_l} \mapsto \omega_\beta\left(\prod_{l=1}^{n} A_{k_l}\right) := \prod_{\{l:k_l<0\}} \mathrm{Tr}_{\mathbb{C}^2}\, \sigma^1 \rho_\beta \sigma^1 A_{k_l} \prod_{\{m:k_m\geqslant 0\}} \mathrm{Tr}_{\mathbb{C}^2}\, \rho_\beta A_{k_m} \tag{4}$$

extends to the unique (α^s, β) KMS state ω_β over $\mathcal{A}$ which is invariant under both time-evolution automorphism groups α and α^s, corresponding to the constant and the soliton external field†. Note that we have put $\rho_\beta = (\mathrm{Tr}\exp(-\beta(M/2)\sigma^3))^{-1}\exp(-\beta(M/2)\sigma^3) = \mathrm{diag}(1/(1+\mathrm{e}^{\beta M}), 1/(1+\mathrm{e}^{-\beta M}))$. As $\beta \to \infty$, ω_β tends in the w^*-topology to the zero-temperature ground state ω_∞ which is defined by

$$\omega_\infty\left(\prod_{l=1}^{n} A_{k_l}\right) = \lim_{\beta\to\infty} \omega_\beta\left(\prod_{l=1}^{n} A_{k_l}\right) = \prod_{\{l:k_l<0\}} (A_{k_l})_{11} \prod_{\{m:k_m\geqslant 0\}} (A_{k_m})_{22}. \tag{5}$$

We find once more $\omega_\infty \circ \alpha_t^s = \omega_\infty \circ \alpha_t = \omega_\infty$, which follows from the preceding definition. By identification of the GNS Hilbert space $\mathcal{H}_\infty$ with the incomplete infinite tensor product $\bigotimes_{k\in\mathbb{Z}}(\mathbb{C}^2)_k$ the cyclic vector Ω_∞ must be identified with the product vector $\bigotimes_{l=-\infty}^{-1}(e_1)_l \otimes \bigotimes_{m=0}^{\infty}(e_2)_m$, where $e_1 = (1,0)^t$, $e_2 = (0,1)^t$. It is clear that non-translational invariance of the soliton external field implies non-invariance of ω_β under translation automorphisms τ_n. Moreover, a translation of the domain wall, say to the right, for n units amounts to switch the direction of all spins in the interval $[0, n-1]$. Therefore the implementer $V_\infty(n)$ of translations, which is unique up to a phase due to irreducibility of the GNS representation associated to ω_∞, has been defined in [1] by first determining its action on the cyclic vector Ω_∞ via

$$V_\infty(n)\Omega_\infty := \Theta(n)\pi_\infty\left(\prod_{k=0}^{n-1}\sigma_j^1\right)\Omega_\infty + \Theta(-n-1)\pi_\infty\left(\prod_{j=-|n|}^{-1}\sigma_j^1\right)\Omega_\infty + \delta_{n0}\Omega_\infty. \tag{6}$$

In order to implement the automorphism τ_n, $V_\infty(n)$ has to act according to

$$V_\infty(n)\pi_\infty(A)\Omega_\infty := \pi_\infty \circ \tau_n(A) V_\infty(n)\Omega_\infty \tag{7}$$

† We remark that τ_n and α_t^s commute when applied to the state ω_β. Due to α_t^s-invariance of ω_β we then have $\omega_\beta \circ \tau_n \circ \alpha_t^s = \omega_\beta \circ \tau_n$, that is α_t^s-invariance of the translated state.

on the total set of vectors $\pi_\infty(A)\Omega_\infty$, $A \in \mathcal{A}$ in the GNS Hilbert space. The time-evolution automorphism groups corresponding to the constant and the soliton external field are implemented on $\mathcal{H}_\infty$ by unitary one-parameter groups U_∞ and U^s_∞, respectively. Whereas the generator of the latter is bounded from below, denoted by H^s_∞, which is the true Hamiltonian of the system in the external soliton field, the generator of the former, denoted by H_∞ has a symmetric spectrum. From

$$\langle V_\infty(n)\Omega_\infty, H_\infty V_\infty(n)\Omega_\infty\rangle = nM \tag{8}$$

we learn that moving the domain wall to the left allows the extraction of an infinite amount of energy out of the system.

Now an anomaly shows up. Although the automorphisms α_t and τ_n commute, the implementers do not. Indeed one finds

$$V_\infty(n)U_\infty(t) = \mathrm{e}^{-\mathrm{i}tMn}U_\infty(t)V_\infty(n). \tag{9}$$

The underlying mechanism causing the anomaly is simply the spectral shift of H_∞ induced by the adjoint action of $V_\infty(n)$. More precisely, $\mathrm{Ad}_{V_\infty(n)}H_\infty = H_\infty - nM\mathbb{1}$.

3. The anomaly at finite T

What happens to this anomaly when the system is put into contact with a thermal bath? First of all we note that, due to ω_β-invariance, the automorphism groups α and α^s are implemented in the reducible GNS representation π_β by unitary one-parameter groups U_β and U^s_β, which are unique once the arbitrary phase has been fixed by the standard convention $U_\beta(t)\Omega_\beta = U^s_\beta(t)\Omega_\beta = \Omega_\beta$. In contrast, the implementers $V_\beta(n)$ of translations are left undetermined up to a unitary element in the commutant $\pi_\beta(\mathcal{A})'$, if they exist at all. However, it turns out that translation symmetry is not spontaneously broken, and there are various choices possible. One of them proposed in [2] uses modular theory to construct implementers $\tilde{V}_\beta(n)$ which act for, say, $n > 0$ on Ω_β according to $\tilde{V}_\beta(n)\Omega_\beta = \pi_\beta(\prod_{j=1}^{n-1}\sigma^1_j)\pi'_\beta(\prod_{j=1}^{n-1}\sigma^1_j)\Omega_\beta$, where $\pi'_\beta(\cdot) = J\pi_\beta(\cdot)J : \mathcal{A} \to \pi_\beta(\mathcal{A})'$ is antilinear and J denotes the modular conjugation. Due to antilinearity of π'_β it happens that the anomaly vanishes since the phase factors cancel. However, the price to pay is that the (*a priori* β-dependent) expectation value

$$\langle \tilde{V}_\beta(n)\Omega_\beta, H^s_\beta \tilde{V}_\beta(n)\Omega_\beta\rangle = 0 \qquad \forall\beta \tag{10}$$

vanishes, and hence does not tend in the $\beta \to \infty$ to the zero-temperature expectation value $\langle V_\infty(n)\Omega_\infty, H^s_\infty V_\infty(n)\Omega_\infty\rangle = |n|M$. Thus choosing $\tilde{V}_\beta(n)$ as implementers would imply that translating the ground state does not cost energy, although one easily calculates that the magnetization changes proportional to $n\tanh(\beta M/2)$.

We propose a different implementation $V_\beta(n)$ which is a straightforward generalization of the zero-temperature case. Let

$$V_\beta(n)\Omega_\beta := \Theta(n)\pi_\beta\left(\prod_{j=0}^{n-1}\sigma^1_j\right)\Omega_\beta + \Theta(-n-1)\pi_\beta\left(\prod_{j=-|n|}^{-1}\sigma^1_j\right)\Omega_\beta + \delta_{n0}\Omega_\beta \tag{11}$$

and extend $V_\beta(n)$ as in (7) on $\mathcal{H}_\beta$, then it is easy to verify that $V_\beta(n)^* = V_\beta(-n) = V_\beta(n)^{-1}$ and $V_\beta(n)V_\beta(m) = V_\beta(n+m)$. Moreover, the energy expectation value of the translated state turns out to be

$$\langle V_\beta(n)\Omega_\beta, H^s_\beta V_\beta(n)\Omega_\beta\rangle = |n| M \tanh\left(\frac{\beta M}{2}\right) \tag{12}$$

which tends in the $\beta \to \infty$ limit to the zero-temperature expectation value $|n|M$ cited above. Note that for both vector states $V_\beta(n)\Omega_\beta$ and $\tilde{V}_\beta(n)\Omega_\beta$ the expectation value of $\pi_\beta(\sigma^3_j)$, which is the spin at site j, equals $\mathrm{sgn}(j)\tanh(\beta M/2)$ and the change in magnetization is equal for both choices. Therefore a translation which amounts to exchanging the probabilities for spin up and spin down in a finite interval also changes the energy of the state and we conclude, due to (10) and (12), that $V_\beta(n)$ is the correct choice.

Remark. To get a better understanding of the two different ways to choose the translation operator, we can also have a look at the finite spin chain of length $2N+1$: there the Hamiltonian h^s, the time-evolution operator $\exp(ith^s)$ and the density matrix $\rho := \exp(-\beta h^s)/Z$, with $Z := \mathrm{Tr}(\exp(-\beta h^s))$ characterizing the kink state, can be written down explicitly as operators acting on $(\mathbb{C}^2)^{2N+1}$; no doubling or GNS construction are needed. The translation operators are represented on $(\mathbb{C}^2)^{2N+1}$ by imposing antiperiodic boundary conditions in the obvious way. Then the operator $V_\beta(n)$ acts on the kink ground state as $\sigma^1_0\sigma^1_1 \ldots \sigma^1_{n-1}$. One then easily calculates for the expectation value of $V_\beta(1)$

$$\langle V_\beta(1)\rangle := \mathrm{Tr}_{(\mathbb{C}^2)^{2N+1}} \rho V_\beta(1) = 0. \tag{13}$$

The translation operator is generated by the lattice momentum and it's expectation values should therefore not be altered by the choice of the doubled implementer. We compute for 'our' choice

$$\langle \Omega_\beta, V_\beta(1)\Omega_\beta\rangle = 0 \tag{14}$$

and get for the other one

$$\langle \Omega_\beta, \tilde{V}_\beta(1)\Omega_\beta\rangle = \sqrt{\frac{2}{1+\cosh\beta M}}. \tag{15}$$

So the first choice is in accordance with what one would expect from the thermodynamic limit of the finite spin-chain. We can also calculate the difference of the energy expectation values of the kink-state and the translated kink-state and get exactly the same result as for the infinitely long spin-chain and 'our' choice of implementer.

Although our choice of $V_\beta(n)$ yields reasonable zero-temperature limits, it is interesting to note that here also $V_\beta(n) \notin \pi_\beta(\mathcal{A})''$. This is seen from

$$V_\beta(n)\pi'_\beta(A)V_\beta(n)^* = \pi'_\beta(\mathrm{Ad}_{\prod_{j=1}^{n-1}\sigma^1_j} \circ \tau_n(A)) \tag{16}$$

($n \geqslant 0$), which can easily be verified on the cyclic and separating vector Ω_β.

In our case the anomaly does not vanish. Instead it takes values in $\pi_\beta(\mathcal{A})'$. One finds for the group commutator

$$V_\beta(n)U_\beta(t)V_\beta(n)^*U_\beta(t)^* = \begin{cases} \pi'_\beta\Big(\prod_{j=0}^{n-1}\exp(-\mathrm{i}tM\sigma_j^3)\Big) & \text{for } n>0 \\ \mathbb{1} & \text{for } n=0 \\ \pi'_\beta\Big(\prod_{j=-|n|}^{-1}\exp(-\mathrm{i}tM\sigma_j^3)\Big) & \text{for } n<0. \end{cases} \tag{17}$$

To see this let $n>0$. We have

$$\begin{aligned}
&\Big(V_\beta(n)U_\beta(t)V_\beta(n)^*U_\beta(t)^* - \pi'_\beta\Big(\prod_{j=0}^{n-1}\exp(-\mathrm{i}tM\sigma_j^3)\Big)\Big)\Omega_\beta \\
&\quad = V_\beta(n)\pi_\beta\Big(\prod_{j=-|n|}^{-1}\exp\Big(\mathrm{i}t\frac{M}{2}\sigma_j^3\Big)\sigma_j^1\exp\Big(-\mathrm{i}t\frac{M}{2}\sigma_j^3\Big)\Big)\Omega_\beta \\
&\qquad - J\pi_\beta\Big(\prod_{j=0}^{n-1}\exp(-\mathrm{i}tM\sigma_j^3)\Big)\Omega_\beta \\
&\quad = \pi_\beta\Big(\prod_{j=0}^{n-1}\exp\Big(\mathrm{i}t\frac{M}{2}\sigma_j^3\Big)\sigma_j^1\exp\Big(-\mathrm{i}t\frac{M}{2}\sigma_j^3\Big)\sigma_j^1\Big)\Omega_\beta \\
&\qquad - \exp\Big(-\frac{\beta}{2}H_\beta^s\Big)\pi_\beta\Big(\prod_{j=0}^{n-1}\exp(-\mathrm{i}tM\sigma_j^3)\Big)^*\Omega_\beta \\
&\quad = \pi_\beta\Big(\prod_{j=0}^{n-1}\exp(\mathrm{i}tM\sigma_j^3)\Big)\Omega_\beta - \pi_\beta\Big(\prod_{j=0}^{n-1}\exp(\mathrm{i}tM\sigma_j^3)\Big)\Omega_\beta \\
&\quad = 0.
\end{aligned}$$

But then (17) follows, since Ω_β is separating for the commutant. Note that we have used the explicit form $\Delta^{1/2} = \exp-(\beta/2)H_\beta$ for the modular operator. The case $n<0$ is treated similarly.

What we have got here is an explicit example of a non-Abelian cohomology class† in the sense of [1]. Denote by (t,n) elements in $U(1)\times\mathbb{Z}$, The group generated by α and τ_n. Define a map $Q_\beta : U(1)\times\mathbb{Z}\to\mathcal{B}(\mathcal{H}_\beta)$ by $Q_\beta(t,n) := U_\beta(t)V_\beta(n)$. Then it is easy to see that

$$Q_\beta(t,n)Q_\beta(t',n') = c_\beta(t,n;t',n')Q_\beta(t+t',n+n') \tag{18}$$

where $c_\beta(t,n;t',n')$ equals the right-hand side of (17), except that t is replaced by t'. Moreover, if we define a map $\eta : U(1)\times\mathbb{Z}\to\mathrm{Aut}(\pi_\beta(\mathcal{A})')$ by $\eta_{(t,n)} := \mathrm{Ad}_{Q_\beta(t,n)}|\pi_\beta(\mathcal{A})'$, then c_β satisfies the modified cocycle identity

$$c_\beta(t_1,n_1;t_2,n_2)c_\beta(t_1+t_2,n_1+n_2;t_3,n_3) = \eta_{(t_1,n_1)}(c_\beta(t_2,n_2))c_\beta(t_1,n_1;t_2+t_3,n_2+n_3) \tag{19}$$

† Although the cocycle in this example takes values in a commutative subgroup of the commutant, we use the expression 'non-Abelian' cohomology class as a technical term.

where η satisfies $\eta_{(t_1,n_1)} \circ \eta_{(t_2,n_2)} = \mathrm{Ad}_{c_\beta(t_1,n_1;t_2,n_2)} \circ \eta_{(t_1+t_2,n_1+n_2)}$. Thus the pair (c_β, η) determines an equivalence class in the sense of [1]. Moreover, it is a remarkable fact of our choice of $V_\beta(n)$ that the $\pi_\beta(\mathcal{A})'$-valued cocycle c_β tends continuously to the phase factor in (9). Although for different β the representations π_β are mutually inequivalent, we are still allowed to take the $\beta \to \infty$ limit of matrix elements between a dense set of vectors in the GNS representation space. Using (4), a straightforward calculation yields

$$\lim_{\beta\to\infty} = \frac{\langle \pi_\beta(A)\Omega_\beta, V_\beta(n)U_\beta(t)V_\beta(n)^* U_\beta(t)^* \pi_\beta(B)\Omega_\beta\rangle}{\langle \pi_\beta(A)\Omega_\beta, \pi_\beta(B)\Omega_\beta\rangle} = \mathrm{e}^{-\mathrm{i}Mnt} \tag{20}$$

whenever $\langle \pi_\beta(A)\Omega_\beta, \pi_\beta(B)\Omega_\beta\rangle \neq 0$ and 0 otherwise. Thus, according to (20) the cocycle $c_\beta(t, n; t', n')$ 'freezes' to the phase factor $\mathrm{e}^{-\mathrm{i}t'nM}$ in the $\beta \to \infty$ limit.

Remark. Note that for the choice $\tilde{V}_\beta(n)$ we may define $\tilde{Q}_\beta(t, n) := U_\beta(t)\tilde{V}_\beta(n)$. Then the multiplier vanishes and we have $Q_\beta(t, n)Q_\beta(t', n') = Q_\beta(t + t', n + n')$, which is referred to in [2] as 'anomaly meltdown'. Therefore our non-commutative cocycle turns out to be a coboundary. To be more precise, put $\mu_\beta(t, n) := Q_\beta(t, n)\tilde{Q}_\beta(t, n)^*$. We have

$$\mu_\beta(t,n) = \begin{cases} \pi'_\beta\left(\prod_{j=0}^{n-1} \exp\left(\mathrm{i}\frac{Mt}{2}\sigma_j^3\right)\sigma_j^1 \exp\left(-\mathrm{i}\frac{Mt}{2}\sigma_j^3\right)\right) & \text{for } n > 0 \\ \mathbb{1} & \text{for } n = 0\,. \\ \pi'_\beta\left(\prod_{j=-|n|}^{-1} \exp\left(\mathrm{i}\frac{Mt}{2}\sigma_j^3\right)\sigma_j^1 \exp\left(-\mathrm{i}\frac{Mt}{2}\sigma_j^3\right)\right) & \text{for } n < 0 \end{cases} \tag{21}$$

A straightforward calculation then shows that the calculation

$$c_\beta(t, n; t', n') = \mu_\beta(t, n)\mathrm{Ad}_{\tilde{Q}_\beta(t,n)}(\mu_\beta(t', n'))\mu_\beta(t + t', n + n')^{-1} \tag{22}$$

is valid. What we learn from this example is the fact that, although the cocycle may be gauged away by redefining $Q_\beta(t, n)$ according to $\tilde{Q}_\beta(t, n) := \mu_\beta(t, n)^{-1}Q_\beta(t, n)$, we are not allowed to do so on physical grounds. From the preceding discussion (see, for instance, (10) and (12)) it is clear that the difference in energy of ω_β and the translated state $\omega_\beta \circ \tau_n$ must come out correctly, which turns out to be the case for the choice of $Q_\beta(t, n)$ only.

In addition we see from (20) that the non-commutative coboundary $c_\beta(t, n; t', n')$ 'freezes' to the phase factor $\mathrm{e}^{-\mathrm{i}Mt'n}$ which determines a non-trivial element in $H^2(U(1) \times \mathbb{Z}, U(1))$. However, this is only possible since the zero-temperature limit of the matrix elements $\langle \pi_\beta(A)\Omega_\beta, \mu_\beta(t, n)\pi_\beta(B)\Omega_\beta\rangle$ does not exist.

4. Local charges in 2D-QFT at finite T

4.1. *Temperature zero*

Let $\mathcal{H}$ be the Hilbert space $\mathcal{L}^2(\mathbb{R}, \mathbb{C}^2)$ and h a selfadjoint Dirac–Hamiltonian on $\mathcal{H}$. Introduce the negative energy projector $P := \Theta(-h)$. Let ω_P be the quasifree, gauge-invariant state on CAR($\mathcal{H}$) defined by its two-point function.

$$\omega_P(a^\dagger(f)a(g)) = \langle f, Pg\rangle. \tag{23}$$

Let $(\mathcal{H}_P, \pi_P, \Omega_P)$ be the associated GNS-triple.

Implement local chiral gauge transformations $\exp(it(\alpha + \alpha_5\gamma_5))$ in $\prod_P$: if we use the free, massive Dirac–Hamiltonian, this puts constraints on the asymptotics of α_5, but not on α [5]. If we demand strong implementability, α_5 must vanish at infinity, whilst α does not have to. As we want to have both functions out of the same class, we take both from $\mathrm{Map}(S^1, \mathrm{Lie}(U(1)))$.

From the condition that $\Gamma_P(\exp(it\alpha)) = \exp(itd\Gamma_P(\alpha))$ implements the gauge transformation, we find (using the abbreviation $a_P(f) := \pi_P(a(f))$)

$$[d\Gamma_P(\alpha), a_P^\dagger(f)] = a_P^\dagger(\alpha f). \tag{24}$$

Since the implementer $d\Gamma_P(\alpha)$ is only defined up to an additive constant, we choose $\langle \Omega_P, d\Gamma_P(\alpha)\Omega_P\rangle = 0$, which is nothing but normal ordering.

We now take $f \in (1-P)\mathcal{H}$ and calculate one-particle expectation values, using that $a_P(f)\Omega_P = 0$:

$$\begin{aligned}\langle a_P^\dagger(f)\Omega_P, d\Gamma_P(\alpha)a_P^\dagger(f)\Omega_P\rangle &= \langle a_P^\dagger(f)\Omega_P, (a_P^\dagger(f)d\Gamma_P(\alpha) + a_P^\dagger(\alpha f))\Omega_P\rangle \\ &= \langle \alpha f, (1-P)f\rangle \\ &= \langle \alpha f, f\rangle.\end{aligned} \tag{25}$$

The local charge operator measures the amount of charge in the region $\mathrm{supp}\alpha \subset \mathbb{R}$ and is an observable. The fact that a Schwinger term arises between local charge and local axial charge

$$[d\Gamma_P(\alpha), d\Gamma_P(\alpha_5\gamma_5)] = \int_{\mathbb{R}} \alpha\alpha_5' \tag{26}$$

is a statement of incommensurability of both quantities.

4.2. $T > 0$

Let ω_T be the state associated with $T = (\exp(\beta h) + 1)^{-1}$ describing the canonical ensemble [6]. The associated GNS representation $(\mathcal{H}_T, \pi_T, \Omega_T)$ is reducible; the vacuum is cyclic and separating. Therefore there is no good particle interpretation: the vacuum is not annihilated by a subset of the CAR.

We implement gauge transformations *à la* Lundberg [3] and get an implementer whose generator is affiliated to $\pi_T(\mathrm{CAR}(\mathcal{H}))''$. Since $\pi_T(\mathrm{CAR}(\mathcal{H}))''$ is a factor, the generator $d\Gamma_T(\alpha)$ is uniquely determined by demanding its vanishing vacuum expectation value. Thus the local gauge transformations act as inner automorphisms of this Von Neumann algebra. We still can calculate matrix elements:

$$\begin{aligned}\langle a_T^\dagger(f)\Omega_T, d\Gamma_T(\alpha)a_T^\dagger(f)\Omega_T\rangle &= \langle a_T^\dagger(f)\Omega_T, (a_T^\dagger(f)d\Gamma_T(\alpha) + a_T^\dagger(\alpha f))\Omega_T\rangle \\ &= \langle a_T(f)a_T^\dagger(f)\Omega_T, d\Gamma_T(\alpha)\Omega_T\rangle + \omega_T(a(f)a^\dagger(\alpha f)).\end{aligned} \tag{27}$$

We should remark explicitly, that $\forall f \in \mathcal{H}$, $a_T(f)a_T^\dagger(f)\Omega_T$ is not a multiple of Ω_T.

Remark. Implementability can easily be shown, as in [4]. One can also show that introducing a chemical potential $\mu \neq 0$ and dealing with the grand canonical ensemble makes no difference for implementability of (axial) gauge transformations with generators with compact support.

When doing the doubling construction, we dont want matrix elements to change. The doubling-up enlarges $\mathcal{H}$ to $\mathcal{H} \oplus \mathcal{H}$ and we investigate the pure state ω_{P_T} over CAR($\mathcal{H}$) defined by the projector

$$P_T := \begin{pmatrix} T & T^{1/2}(1+T)^{1/2} \\ T^{1/2}(1-T)^{1/2} & 1-T \end{pmatrix}. \tag{28}$$

There is an ambiguity in lifting the Bogoliubov automorphisms of CAR($\mathcal{H}$) to Bogoliubov automorphisms of CAR($\mathcal{H} \oplus \mathcal{H}$). Let $\iota : \mathcal{U}(\mathcal{H}) \to \mathcal{U}((\mathcal{H})$ to be a homomorphism of unitary groups. Then it is clear, that the unitary $U \oplus \iota(U)$ induces a Bogoliubov automorphism on CAR($\mathcal{H} \oplus \mathcal{H}$) which restricts to the automorphism induced by U on CAR($\mathcal{H} \oplus 0$). Hence *the generators of local gauge transformations are not unique in this scheme, once* ι *has not* been fixed. Denoting by abuse of notation the derivative of ι by the same letter, we may write this generator as $d\Gamma_{P_T}(\alpha \oplus \iota(\alpha))$. We will concentrate on the two possibilities $\iota \equiv 0$ and $\iota \equiv id$ for the derivative. From $d\Gamma_{P_T}$ being an implementer, we deduce

$$[d\Gamma_{P_T}(\alpha \oplus \iota(\alpha)), a^{\dagger}_{P_T}(f \oplus g)] = a^{\dagger}_{P_T}(\alpha f \oplus \iota(\alpha) g). \tag{29}$$

• The choice $\iota \equiv 0$. This corresponds to $d\Gamma_{P_T}(\alpha \oplus 0)$ affiliated to $\pi_{P_T}(\mathrm{CAR}(\mathcal{H} \oplus 0)''$. This follows from (29) together with $\pi_{P_T}(\mathrm{CAR}(\mathcal{H} \oplus 0))' = \mathrm{e}^{\mathrm{j}\pi F} \pi_{P_T}(\mathrm{CAR}(0 \oplus \mathcal{H}))''$, where F is the fermion number operator. Note also that $d\Gamma_{P_T}(\alpha \oplus 0)$ corresponds exactly to the choice of $d\Gamma_T(\alpha)$. Since local charge is a measurable quantity, the unitary implementers should be reached by strong limits from the algebra.

• The choice $\iota \equiv id$. Although for this choice the matrix elements of $d\Gamma_{P_T}(\alpha \oplus \alpha)$ taken between vectors $a^{\dagger}_{P_T}(f_1 \oplus 0) \ldots \Omega_{P_T}$ are the same as the matrix elements of $d\Gamma_{P_T}(\alpha \oplus 0)$, the former generator is no more affiliated. Moreover, as has been noted in a previous publication [4], the choice $\iota \equiv id$ (which equals the diagonal embedding $U \mapsto U \oplus U$ for Bogoliubov automorphisms), kills the Schwinger term. Thus there is a mechanism which allows us to eliminate the projective multiplier, which occurs due to the normal ordering procedure.

Using the choice $\iota \equiv id$, one easily deduces from non-invariance of P_T under conjugation with $\mathrm{e}^{\mathrm{i}\lambda\alpha} \oplus \mathrm{e}^{\mathrm{i}\lambda\alpha}$ that $\mathrm{e}^{\mathrm{i}\lambda d\Gamma_{P_T}(\alpha \oplus \alpha)}$ does not leave the cyclic vector invariant.

If we insist on treating the algebra elements and local charges on the same footing, we may calculate the same matrix element as before. With $\iota(f) := 0$ or f whenever ι is 0 or id we find

$$\begin{aligned}\langle a^{\dagger}_{P_T}(f \oplus \iota(f))\Omega_{P_T}, d\Gamma_{P_T}(\alpha \oplus \iota(\alpha)) a^{\dagger}_{P_T}(f \oplus \iota(f))\Omega_{P_T}\rangle \\ = \langle a_{P_T}(f \oplus \iota(f)) a^{\dagger}_{P_T}(f \oplus \iota(f))\Omega_{P_T}, d\Gamma_{P_T}(\alpha \oplus \iota(\alpha))\Omega_{P_T}\rangle \\ + \omega_{P_T}(a_{P_T}(f \oplus \iota(f)) a^{\dagger}_{P_T}(\alpha f \oplus \iota(\alpha)\iota(f))).\end{aligned} \tag{30}$$

Since we demand matrix elements to be identical, we can now compare

$$\omega_T(a_T(f) a^{\dagger}_T(\alpha f)) = \langle \alpha f, (1-T) f\rangle \tag{31}$$

with

$$\begin{aligned}\omega_{P_T}(a_{P_T}(f\oplus\iota(f))a^\dagger_{P_T}(\alpha f\oplus\iota(\alpha)\iota(f))) &= \langle\alpha f\oplus\iota(\alpha)\iota(f),(1-P_T)f\oplus\iota(f)\rangle\\ &= \langle\alpha f,(1-T)f\rangle+\langle\iota(\alpha)\iota(f),T\iota(f)\rangle-\langle\alpha f,\sqrt{T(1-T)}\iota(f)\rangle\\ &\quad -\langle\iota(\alpha)\iota(f),\sqrt{T(1-T)}f\rangle. \end{aligned} \tag{32}$$

We see that the only possible choice is $\iota\equiv 0$.

5. Global charges in 2D-QFT at finite T

At finite temperature it turns out that global quantities like charge and axial charge are no more affiliated to the observable algebra, in contrast to local operations. Consider, for example, rigid gauge transformations $e^{i\alpha}$, $\alpha\in\mathbb{R}$. Since ω_T, $T=(\exp(\beta h)+1)^{-1}$ is left invariant by the gauge automorphism $a^\dagger(f)\mapsto a^\dagger(e^{i\alpha}f)$; the gauge symmetry is *implemented by* $e^{i\alpha Q}$ *on the* GNS *space. Now it is easy to see that* $e^{i\alpha Q}$ *cannot commute* with operators in the commutant. From

$$(e^{i\alpha Q}\pi'_T(a^\dagger(f))e^{-i\alpha Q}-\pi'_T(a^\dagger(e^{i\alpha}f)))\Omega_{P_T}=0 \tag{33}$$

we learn that $e^{i\alpha Q}$ induces the same gauge automorphism on the commutant, and therefore cannot be affiliated with $\pi_T(\mathrm{CAR}(\mathcal{H}))''$. Another way to see this is to check the Hilbert–Schmidt norm of $P_T-(e^{i\alpha}\oplus\mathbb{1})P_T(e^{i\alpha}\oplus\mathbb{1})^{-1}$, which can easily be seen to be infinite for non-compact space $\mathbb{R}$. Thus rigid gauge transformations do not act as inner automorphisms of the Von Neumann algebra, and we conclude that global quantities at finite temperature become physically troublesome.

Acknowledgment

We would like to thank Professor Streater for a very interesting correspondence on the topics dealt with in this paper.

References

[1] Streater R F 1990 *Commun. Math. Phys.* **132** 201–15
[2] Gajdzinski C and Streater R F 1991 *J. Math. Phys.* **32** 1981
[3] Lundberg L E 1976 *Commun. Math. Phys.* **50** 103
[4] Grosse H, Maderner W and Reitberger C 1993 *J. Math. Phys.* **34** 4469
[5] Gallone F, Sparzani A, Streater R F and Ubertone C 1986 *J. Phys. A: Math. Gen.* **19** 241
[6] Bratteli O and Robinson D W 1981 *Operator Algebras and Quantum Statistical Mechanics* vol II (Berlin: Springer)